length analysis of millimeter wavelength sources too high in frequency to be measured conventionally.

Operating in the TEM mode the interferometer represents the ideal form of cavity resonator, permitting the use of relatively large structures at very small wavelengths with complete freedom from troubles due to higher order modes. The Q values obtained here are higher than can readily be attained by a conventional cavity resonator at these frequencies, and the indications are that still higher Q values can be obtained for apertures and reflectors larger in terms of the wavelength. As the cavity resonator for ultramicrowaves, the use of the interferometer for dielectric constant and loss measurements on both solids and gases is clearly indicated, and in the ultramicrowave region of the spectrum such a method becomes most advantagoues. Its use in all other microwave devices employing a cavity is also possible, and in particular it would appear that the interferometer can be used with facility as the cavity resonator for masers designed to operate at millimeter and submillimeter wavelengths.

Aperture and reflector dimensions of 24 λ and 50 λ, respectively, in extent were used here, and the insertion loss at optimum transmission is around 15 db. Consequently, the reflected power is quite high so that the fringes are best observed in transmission. For smaller wavelengths the problem of adequate aperture and reflector dimensions in terms of the wavelength becomes easier, and interferometers for specific purposes such as maser cavities become easier to accommodate and use with the associated apparatus. Also with apertures and reflectors which are large in terms of the wavelength, the problem of the diffraction correction becomes less severe and important, and the ease with which the interferometer can be used for precision measurements such as the velocity of light, and as a microwave standard of length, will improve with the use of shorter wavelengths.

VI. Acknowledgment

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Boundary Conditions and Ohmic Losses in Conducting Wedges*

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Summary—**The present work is concerned with the boundary conditions required to calculate the ohmic losses occurring in metallic wedges under the influence of electromagnetic waves which are sinusoidal in time. The validity of the surface impedance condition used in calculating waveguide wall losses is examined carefully, and a "modified" surface impedance condition, which can be applied to wedge problems in which the perfectly conducting solution is known, is developed. A simple waveguide having a circular cross section, a sector of which is occupied by a metal wedge, is used as an example. The tangential magnetic field variations along the surface of the wedge are shown graphically, demonstrating, near the tip of the wedge, a large deviation from the tangential magnetic field of the perfectly conducting solution.**

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I. Introduction

THE heat losses within any conducting object caused by the presence of an electromagnetic field, can be calculated by calculating the average flow of power into the object as a result of the tangential fields on its surface. The boundary conditions which must be imposed on the surface of a metallic wedge in order to calculate this power flow must be considered very carefully. The standard surface impedance condition used in the calculation of waveguide wall losses relates the tangential electric field at a conducting boundary to the known tangential magnetic field which would exist at the boundary if it were perfectly conducting. This condition, when applied to wedge problems, often leads to fields which do not satisfy the Meixner edge condition [1] and to infinite power losses in the region of the tip.

Wedge problems have received considerable attention in the past because of the possibility of infinite field strengths at the tip of any wedge-shaped boundary. Because of this, the common technique of discarding wave functions which possess singularities cannot be used, and many boundary value problems involving wedges appear at a first glance to lack uniqueness. In 1949 Meixner published his classical paper [1] on the "edge condition," which added a further "boundary condition" to wedge problems, making their solution, in any situation, unique. Since then, the formulation of problems involving perfectly conducting wedges has been quite straightforward, although the subsequent solutions to many such problems are very complex. In recent years, the problem of diffraction by an imperfectly conducting wedge has been treated [2], [3] and the surface impedance boundary condition used. Although some doubt was expressed about the validity of the condition in the neighborhood of the wedge tip, it was assumed to hold for the entire wedge face and no apparent difficulties resulted.

Difficulties do arise, however, when the surface impedance condition is used to modify solutions to perfectly conducting wedge problems for the purpose of calculating ohmic losses, and, for this reason, the general problem of field behavior near conducting wedges is studied in the present work. In Section II, the exact field behavior within a few skin depths of the wedge tip is examined. In Section III, the surface impedance condition is derived using a wedge-shaped boundary and is expressed in an integral form which reduces to the well known surface impedance condition [4] except within a few skin depths of the tip of the wedge. In Section IV, the coupled modes in a wedge or septate waveguide (Fig. 2) are developed and the reason for the apparent breakdown in the surface impedance condition at the tip of the wedge is illustrated. An approximate solution to the problem is developed which agrees with the analyses of Sections II and III and a numerical example, using the wedge waveguide, is presented graphically in Fig. 3.

II. Field Behavior Near the Tip

In treating the behavior of the fields near the tip of the wedge, the cylindrical coordinates r, ϕ, and z are used. The z axis is taken along the axis of the wedge which is shown in cross section in Fig. 1. The wedge has been symmetrically placed with respect to the x axis and, in the region outside of the wedge, the coordinate angle ϕ is measured from the positive x axis. The fields within the metal wedge are represented in terms of another cylindrical coordinate system $\bar{r}$, $\bar{\phi}$, and $\bar{z}$ in which the coordinate angle $\bar{\phi}$ is measured from the negative x axis, as shown. The two faces of the wedge are given by

$$\phi = +\phi_1, \qquad \bar{\phi} = -\phi_2$$

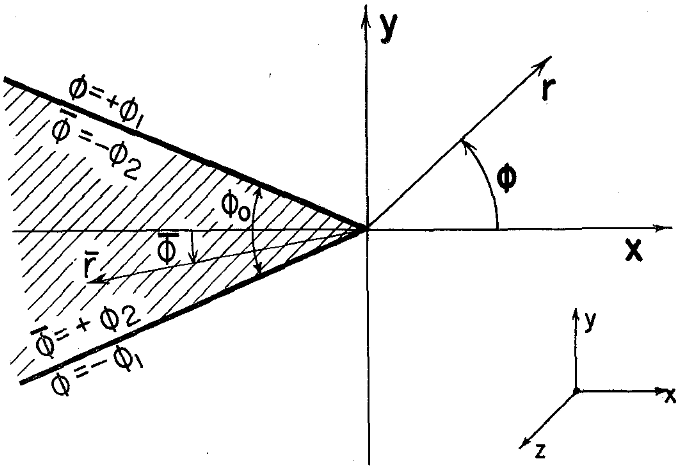

Fig. 1—The coordinate system used.

and

$$\phi = -\phi_1, \qquad \bar{\phi} = +\phi_2,$$

and the total wedge angle ϕ_0 is given by

$$\phi_0 = 2\phi_2.$$

The fields are assumed to vary as exp $(j\omega t)$ and the electric properties of the wedge are designated by a complex permittivity,

$$\bar{\epsilon} = \epsilon_0 - j\sigma/\omega,$$

where $j = \sqrt{-1}$, and σ is the electrical conductivity in mhos per meter. The wedge is assumed to have a magnetic permeability, $\bar{\mu}$, which is very close to μ_0. Variation in the z direction is assumed to be of the form exp $(-j\beta z)$, as it would be in a wedge waveguide.

The other four field components can be expressed in terms of E_z and H_z by the following relationships [5]:

$$E_r = -(j\beta/\alpha^2)\frac{\partial E_z}{\partial r} - (j\omega\mu/\alpha^2 r)\frac{\partial H_z}{\partial \phi}, \tag{1}$$

$$E_\phi = -(j\beta/\alpha^2 r)\frac{\partial E_z}{\partial \phi} + (j\omega\mu/\alpha^2)\frac{\partial H_z}{\partial r}, \tag{2}$$

$$H_r = (j\omega\epsilon/\alpha^2 r)\frac{\partial E_z}{\partial \phi} - (j\beta/\alpha^2)\frac{\partial H_z}{\partial r}, \tag{3}$$

$$H_\phi = -(j\omega\epsilon/\alpha^2)\frac{\partial E_z}{\partial r} - (j\beta/\alpha^2 r)\frac{\partial H_z}{\partial \phi}, \tag{4}$$

where β, in the case of the wedge waveguide, is the axial propagation constant $2\pi/\lambda_G$ and α is related to the free-space propagation constant $k = 2\pi/\lambda_0$, by the relation,

$$\alpha^2 = k^2 - \beta^2.$$

Representations of the form (1) to (4) are valid both inside and outside of the wedge when the appropriate ϵ and μ are used. The constant β must be the same for both regions, but $\bar{\alpha}$ in the interior region is many times larger than α in the exterior region. In both regions, both E_z and H_z must satisfy the two-dimensional wave equation,

$$r\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{\partial^2\psi}{\partial\phi^2}+(\alpha r)^2\psi=0, \qquad (5)$$

where $\psi=E_z$ or H_z and $\bar{\alpha}$ replaces α in the interior region.

Following a procedure similar to that used by Meixner [6], the fields E_z and H_z can be expanded in a power series in r about the origin, $r=0$.

$$E_z=\sum_n c_n r^{t+n-1} \qquad (6)$$

and

$$H_z=\sum_n \gamma_n r^{t+n-1}, \qquad (7)$$

where the c_n's and γ_n's are functions of the angle ϕ and $0<\mathrm{Re}\,(t)<1$. Using (1) to (4), similar series can be derived in terms of the c_n's and γ_n's for the four transverse field components. The Meixner edge condition [1], which requires that the energy density in the fields be integrable at the wedge tip, limits the behavior of the six field components for small values of r. It is a simple matter to show that finite energy requires the two field components parallel to the wedge axis to remain finite as r approaches zero. This, in turn, means that, if $0<\mathrm{Re}\,(t)\leq 1$, c_1 and γ_1 are the smallest nonzero coefficients in the series for E_z and H_z. The transverse field components, however, may behave as r^{t-1} as long as the real part of t is greater than zero.

Substituting (6) and (7) into the wave equation (5), yields simple, ordinary differential equations for the coefficients c_n and γ_n, and in particular,

$$\frac{d^2c_1}{d\phi^2}+t^2c_1=0 \qquad (8)$$

with similar equations for γ_1, $\bar{c}_1$, and $\bar{\gamma}_1$, where the bar indicates fields within the wedge. This means that

$$c_1=l_1\cos(t\phi)+l_2\sin(t\phi), \qquad (9)$$

and

$$\gamma_1=\lambda_1\cos(t\phi)+\lambda_2\sin(t\phi). \qquad (10)$$

Within the metal wedge, $\bar{E}_z$ and $\bar{H}_z$ can be expressed in the same form, and in both regions the behavior of the transverse field components as r approaches zero can be found using (1) to (4) together with (9) and (10). In the outside region,

$$E_r=[L_1\cos(t\phi)+L_2\sin(t\phi)]r^{t-1} \qquad (11)$$

$$E_\phi=[L_2\,\mathrm{cost}\,(t\phi)-L_1\sin(t\phi)]r^{t-1}, \qquad (12)$$

$$H_r=[\Lambda_1\cos(t\phi)+\Lambda_2\sin(t\phi)]r^{t-1}, \qquad (13)$$

$$H_\phi=[\Lambda_2\cos(t\phi)-\Lambda_1\sin(t\phi)]r^{t-1}, \qquad (14)$$

with a similar representation within the wedge. In terms of c_1 and γ_1,

$$L_1=-(jt/\alpha^2)\{\beta l_1+\omega\mu\lambda_2\}, \qquad (15)$$

$$L_2=-(jt/\alpha^2)\{\beta l_2-\omega\mu\lambda_1\}, \qquad (16)$$

$$\Lambda_1=-(jt/\alpha^2)\{\beta\lambda_1-\omega\epsilon l_2\}, \qquad (17)$$

$$\Lambda_2=-(jt/\alpha^2)\{\beta\lambda_2+\omega\epsilon l_1\}. \qquad (18)$$

An identical set of relationships holds for the barred quantities which represent the fields within the wedge. Continuity conditions at the wedge faces can then be used to determine the L's and Λ's in terms of a single amplitude factor. An estimate of how small r must be before the first term is predominant in all of the series involved, can be made by studying (5). This equation can be written in terms of the dimensionless variable (αr) and power series solutions appropriate to each region would involve either (αr) or $(\bar{\alpha}r)$. Since, within the metal, $\bar{\alpha}\simeq(-j\omega\mu\sigma)^{1/2}$, the condition that the first term of the series *within the wedge* be predominant is that $(r/\delta)\ll 1$ where $\delta=(2/\omega\mu\sigma)^{1/2}$ is the skin depth of the metal.

The continuity of E_r, ϵE_ϕ, H_r and μH_ϕ at the two wedge faces yields eight homogeneous, linear equations in the eight unknown coefficients which determine the field behavior near the tip of the wedge. Because of symmetry with respect to the angle ϕ, these equations can be separated into two sets of four equations each. For fields which are even in E_z and odd in H_z with respect to ϕ $(l_2=\bar{l}_2=\lambda_1=\bar{\lambda}_1=0)$,

$$L_1\cos(t\phi_1)-\bar{L}_1\cos(t\phi_2)=0, \qquad (19)$$

$$\epsilon_0L_1\sin(t\phi_1)+\bar{\epsilon}\bar{L}_1\sin(t\phi_2)=0, \qquad (20)$$

$$\Lambda_2\sin(t\phi_1)+\bar{\Lambda}_2\sin(t\phi_2)=0, \qquad (21)$$

$$\mu_0\Lambda_2\cos(t\phi_1)-\bar{\mu}\bar{\Lambda}_2\cos(t\phi_2)=0. \qquad (22)$$

A nontrivial solution to this set of equations exists if, and only if,

$$\begin{vmatrix}\cos(t\phi_1), & -\cos(t\phi_2)\\ \sin(t\phi_1), & g\sin(t\phi_2)\end{vmatrix}\begin{vmatrix}\sin(t\phi_1), & \sin(t\phi_2)\\ \cos(t\phi_1), & -q\cos(t\phi_2)\end{vmatrix}=0, \qquad (23)$$

where

$$g=\bar{\epsilon}/\epsilon_0 \quad\text{and}\quad q=\bar{\mu}/\mu_0.$$

Eq. (23) yields two sets of discrete *eigenvalues* for the exponent t. One set, which depends only on g, makes the first factor in (23) vanish. The other set, which depends only on q, makes the second factor vanish.

A similar set of equations results from fields which are odd in E_z and even in H_z with respect to the angle ϕ in Fig. 1. In general, four sets of eigenvalues for t occur. Two sets depend only on the electric properties of the wedge and are given by the roots of

$$1-g=\frac{\sin(t\pi)}{\cos(t\phi_1)\sin(t\phi_2)}, \qquad (24)$$

and

$$1 - g = \frac{\sin(t\pi)}{\cos(t\phi_2)\sin(t\phi_1)}\cdot \tag{25}$$

The first of these is associated with fields which are even in E_z and the second with fields which are odd in E_z with respect to the coordinate angle ϕ. Replacing g by q in (24) and (25), two other sets of eigenvalues occur which depend only on the magnetic properties of the wedge.

Solutions associated with values of t, determined by the electric properties of the wedge, yield nonzero values for L_1 and $\overline{L}_1$ in (19) to (22), but admit only the trivial solution, $\Lambda_2 = \overline{\Lambda}_2 = 0$, for the coefficients associated with H_r and H_ϕ in (13) and (14). Solutions associated with values of t determined by the magnetic properties of the wedge, on the other hand, yield nonzero values for the Λ's and admit only the trivial solution for the L's. It can be seen, from (11) to (14), that a field component can become infinite at the tip of the wedge *only* if an eigenvalue of t which is less than unity admits a non-zero value for the L's or Λ's associated with that component. This, in turn, means that any singularities in the transverse components of the magnetic field at the tip of the wedge depend only on the magnetic properties of the wedge. Singularities in the transverse components of the electric field, moreover, depend only on the electric properties of the wedge. If the wedge has a magnetic permeability $\bar{\mu}$ equal to μ_0 and a conductivity of σ mhos per meter, then

$$1 - g = (j\sigma/\omega\epsilon_0)$$

and

$$1 - q = 0.$$

The "magnetic eigenvalues" are determined by the roots

$$0 = \frac{\sin(t\pi)}{\cos(t\phi_1)\sin(t\phi_2)} \tag{26}$$

and

$$0 = \frac{\sin(t\pi)}{\cos(t\phi_2)\sin(t\phi_1)} \tag{27}$$

which yield only integral, nonzero values for t. This means that no singularity can occur in either H_r or H_ϕ at the tip of the wedge. If $\bar{\mu}$ is close to μ_0, moreover, t will lie very close to a positive integer and any singularity in H_r or H_ϕ will be of a very low order.

The "electric eigenvalues," on the other hand, are determined by the roots of

$$(j\sigma/\omega\epsilon_0) = \frac{\sin(t\pi)}{\cos(t\phi_1)\sin(t\phi_2)} \tag{28}$$

and

$$(j\sigma/\omega\epsilon_0) = \frac{\sin(t\pi)}{\cos(t\phi_2)\sin(t\phi_1)}\cdot \tag{29}$$

For large values of $(\sigma/\omega\epsilon_0)$, these eigenvalues lie close to the zeros of $\sin(t\phi_1)$, $\sin(t\phi_2)$, $\cos(t\phi_1)$, and $\cos(t\phi_2)$, which are given by

$$t_{1,n} = (n+1)\pi/(2\phi_1) \tag{30}$$

and

$$t_{2,n} = (n+1)\pi/(2\phi_2) \tag{31}$$

where n is a positive integer or zero. Eigenvalues which are even multiples of $\pi/2\phi_1$ or odd multiples of $\pi/2\phi_2$ are associated with solutions for E_z which are odd with respect to the angle ϕ. The others are associated with fields which are even in E_z.

If ϕ_2 lies between 0 and $\pi/2$, then the only eigenvalue less than unity in either (30) or (31) is

$$t_{1,0} = \pi/2\phi_1. \tag{32}$$

For finite but large σ, moreover, the lowest eigenvalue for the metallic wedge is approximately given by

$$t_E = t_{1,0} - \frac{j\omega\epsilon_0 \tan(t_{1,0}\pi)}{\sigma\phi_1}\cdot \tag{33}$$

As σ becomes infinite, t_E approaches $t_{1,0}$, the eigenvalue for the perfectly conducting wedge, as $1/\sigma$ or as δ^2, where δ is the *skin depth*. This "perturbation" in t caused by the finite wall conductivity, therefore, is a second order effect compared to the "coupled modes" discussed in Section III.

The axial fields associated with this lowest eigenvalue are given, near the tip of the wedge, by

$$E_z = l_1 \cos(t_E\phi) r^{t_E} \tag{34}$$

and

$$H_z = \lambda_2 \sin(t_E\phi) r^{t_E}, \tag{35}$$

where l_1 and λ_2 are related to each other from (18) by

$$l_1 = (-\beta/\omega\epsilon_0)\lambda_2 \tag{36}$$

since $\Lambda_2 = 0$. Eq. (36) shows a fixed coupling between E_z and H_z which is independent of the conductivity σ of the wedge. The two magnetic field components which are perpendicular to the axis of the wedge, moreover, cannot have a singularity of an order greater than $r^{t_H - 1}$ where t_H is the lowest eigenvalue associated with the magnetic properties of the wedge. This eigenvalue in practice is very close to unity. Using (3) and (4), these conditions can be expressed mathematically in terms of the two tip equations:

$$\frac{\partial E_z}{\partial\phi} - \frac{\beta}{\omega\epsilon_0} r \frac{\partial H_z}{\partial r} \to 0 \quad \text{faster than } r^{t_H} \tag{37}$$

and

$$\frac{\partial H_z}{\partial\phi} + \frac{\omega\epsilon_0}{\beta} r \frac{\partial E_z}{\partial r} \to 0 \quad \text{faster than } r^{t_H} \tag{38}$$

These equations do not, of course, restrict the behavior of E_z and H_z unless these field components individually approach zero more slowly than r^{t_H}. It can be shown, however, that (37) implies (38) and is a necessary and sufficient condition for finding the coupled E_z near the tip of an imperfectly conducting wedge if H_z is a given wave function which approaches zero as r^t where $t<t_H$ [and indeed t can be as low as t_E in (33)].

III. Field Behavior away from the Tip

In Appendix I an expression is derived for the relationship between E_z and H_r on the surface of a metal wedge where the coordinates used are shown in Fig. 1. The expression is in integral form and holds everywhere on the faces of the wedge. For highly conducting wedges the expression reduces to

$$E_z(r, \phi_1) = \frac{\omega\mu}{2}\int_{-\infty}^{r} H_r H_0^{(2)}(\bar{\alpha}\,|\,x\,|\,)dx \qquad (39)$$

in which E_z is the axial electric field on the surface $\phi=+\phi_1$, a distance r from the tip of the wedge. H_r is the radial magnetic field a distance x from the point at which E_z is being evaluated. $H_0^{(2)}(\bar{\alpha}|x|)$ is a Hankel function of the second kind [7] which behaves as $(\frac{1}{2}\pi\bar{\alpha}|x|)^{-1/2}\exp(-j\bar{\alpha}|x|-\frac{1}{4}\pi)$ for large values of $|x|$. If $\bar{\alpha}$ has a negative imaginary part $H_0^{(2)}$, $(\bar{\alpha}|x|)$ vanishes rapidly as $|x|$ increases and, for r more than a few skin depths, the upper limit in (39) can be taken as $+\infty$. Furthermore, if H_r is continuous and slowly varying near the point $x=0$, it can be replaced by its value at $x=0$, in which case (39) becomes

$$E_z(r, \phi_1) = \frac{\omega\mu}{2} H_r(r, \phi_1)\int_{-\infty}^{\infty} H_0^{(2)}(\bar{\alpha}\,|\,x\,|\,)dx. \qquad (40)$$

This integral can readily be evaluated by using tables of Laplace transforms [8] to yield

$$E_z(r, \phi_1) = \frac{\omega\mu}{2} H_r(r, \phi_1)\frac{2}{(\bar{\alpha})} = Z_s H_r(r, \phi_1) \qquad (41)$$

for large values of wedge conductivity. Z_s is called the surface impedance of the metal and is related to the skin depth δ by

$$Z_s = \frac{\omega\mu\delta(1+j)}{2}\,. \qquad (42)$$

The same analysis can be used to relate E_r at the wedge face $\phi=\phi_1$ to H_z yielding the result,

$$E_r(r, \phi_1) = -Z_s H_z(r, \phi_1). \qquad (43)$$

Relationships on the other wedge face $\phi=-\phi_1$ are identical except for sign.

This *surface impedance condition* relates the tangential electric field at the surface of the wedge to the *total* tangential magnetic field existing at the same point. This relationship is valid to within a few skin depths of the wedge tip, and the error introduced by assuming that it holds over the entire wedge face does not appear to be too large. This assumption has been used successfully in diffraction problems [2] and the integral expression (39) does not indicate any violent breakdown in the relationship. If the actual H_r, however, is assumed to be the same as it would be if the wedge were perfectly conducting, this field becomes infinite at the wedge tip in many problems. Applying the surface impedance condition in this case would yield an axial electric field which would become infinite at the wedge tip in violation of the edge condition. This difficulty, however, can be resolved by considering a simple example.

The lowest propagating mode of the wedge waveguide shown in cross section in Fig. 2, when perfectly conducting walls are assumed, is governed by the axial magnetic field,

$$H_z^0 = J_t(\alpha r)\sin(t\phi) \qquad (44)$$

where $t=t_{1,0}=\pi/2\phi_1$, and $J_t(\alpha r)$ is a Bessel function of the first kind. The constant α is defined in Section II following (4). The radial magnetic field associated with this mode is given by (3), namely,

$$H_r^0 = \frac{-j\beta}{\alpha} J_t'(\alpha r)\sin(t\phi). \qquad (45)$$

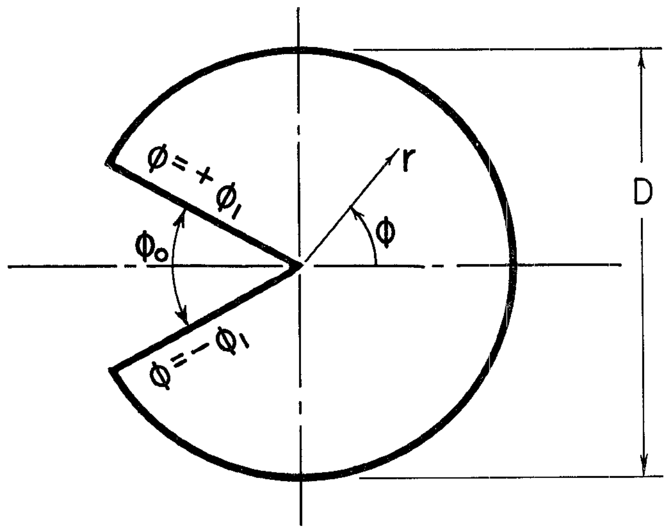

Fig. 2—The wedge waveguide.

The finite conductivity of the walls will introduce an axial electric field proportional to Z_s or to the skin depth δ. This perturbation cannot be taken care of by the small perturbation in the eigenvalue t, which by (33) is proportional to δ^2, and a coupled mode must be introduced. By using a well known relationship for Bessel functions [9] a wave function E_z^e can be found which, on both wedge faces, satisfies the condition,

$$E_z^e = \pm Z_s H_r^0, \qquad (46)$$

and which has the form,

$$E_z^e = -\frac{Z_s j\beta}{2\alpha}\left[\frac{J_{t-1}(\alpha r)\cos[(t-1)\phi]}{\cos[(t-1)\phi_1]} - \frac{J_{t+1}(\alpha r)\cos[(t+1)\phi]}{\cos[(t+1)\phi_1]}\right]. \quad (47)$$

This "coupled" mode, moreover, carries with it a radial magnetic field which is related to E_z^e by (3). Calling this coupled magnetic field H_r^e, it is given by

$$H_r^e = \frac{j\omega\epsilon}{\alpha^2 r}\frac{\partial E_z^e}{\partial\phi}\cdot \quad (48)$$

Since, for any fixed position on the surface of the wedge, E_z^e is proportional to $Z_sH_r^0$, H_r^e is also proportional to $Z_sH_r^0$ and vanishes as the wall conductivity becomes infinite. One is therefore tempted to assume that H_r^e is negligibly small compared to H_r^0 in problems involving metal wedges. The r in the denominator of (48), however, makes H_r^e become arbitrarily large for small values of r, and the ratio of H_r^e to H_r^0 on the wedge surface is given by

$$\frac{H_r^e}{H_r^0} = \frac{-bZ_s}{r}\cot(\phi_1)\left[\frac{t^2J_t(\alpha r)}{(\alpha r)J_t'(\alpha r)} - 1\right] \quad (49)$$

where $b=(j\omega\epsilon/\alpha^2)$ and $Z_s=\omega\mu\delta(1+j)/2$, where δ is the *skin-depth.* This ratio is small if r/δ is large but, for finite values of δ, this ratio becomes infinite as r approaches zero.

This result suggests a modified surface impedance condition in which H_r^0 is replaced by $(H_r^0+H_r^e)$. The surface impedance condition then becomes

$$E_z^e = Z_s[H_r^0 + (j\omega\epsilon/\alpha^2 r)\partial E_z^e/\partial\phi]. \quad (50)$$

For points more than a few skin depths away from the tip of the wedge, (50) is just (46), the surface impedance condition used in all waveguide problems. As r approaches zero, however, (50), which can be rewritten in the form,

$$\frac{r}{bZ_s}E_z^e = \frac{-\beta}{\omega\epsilon}r\frac{\partial H_z^0}{\partial r} + \frac{\partial E_z^e}{\partial\phi}, \quad (51)$$

using (45), approaches the tip equation (37) provided that E_z^e and H_z^0 both approach zero as r^t where $t+1$ is greater than t_H. This will always be the case if E_z^e and H_z^0 satisfy the Meixner edge condition which places a lower limit on the exponent t.

This analysis shows that if the modified surface impedance condition is assumed to hold over the entire wedge face, fields which are in agreement with the analysis of Section II can be found. These fields will be identical with those derived from (44) and (47) for large values of r/δ and the breakdown will occur when H_r^e, given by (48), becomes comparable in magnitude to H_r^0 given by (45). As r/δ approaches zero, moreover, the radial component of the magnitic field will approach zero as shown in Section II.

IV. Finding Approximate Solutions

In Appendix II it is shown that, if H_z^0 is a given wave function which satisfies the Meixner edge condition at the tip of the wedge, then any function E_z which satisfies the tip equation (37) will also satisfy the wave equation for small values of r, and will give rise to fields which are in keeping with the "tip" analysis given in Section II, provided only that none of the resulting field components become independent of the angle ϕ in Fig. 1, as r approaches zero. This condition can usually be satisfied by inspection when looking for approximate solutions. Eq. (37), moreover, can be written in the form,

$$\frac{\partial E_z}{\partial\phi} + (r/b)H_r^0 + O(r^x) = 0, \quad (52)$$

where $b=(j\omega\epsilon/\alpha^2)$, $H_r^0 = -(j\beta/\alpha^2)\partial H_z^0/\partial r$, and $O(r^x)$ is any function which approaches 0 as r^x where $x>t_H$. Away from the tip region the coupled field can be found by ordinary methods since its value on the faces of the wedge and on other boundaries is a known quantity related to the tangential magnetic field of the perfectly conducting solution to the problem. For large values of r/δ, therefore, E_z is known and has, in the simplest case, the form,

$$E_z = Z_sH_r^0/P(\phi), \quad (53)$$

where $P(\phi)$ is a well behaved, odd function of the angle ϕ such that $P(\phi_1)=1$. If (53) is true, then a solution of

$$\frac{\partial E_z}{\partial\phi} - \frac{r}{bZ_s}P(\phi)E_z = -\frac{r}{b}H_r^0 \quad (54)$$

has the following properties:

1) For small values of $|r/b\ Z_s|$ it satisfies (52).
2) For large values of $|r/b\ Z_s|$ it satisfies (53) and therefore approaches its correct value away from the tip region,
3) For all values of r it satisfies the surface impedance condition on the faces of the wedge.

Functions which satisfy (54) can be found quite easily; the general solution has the form,

$$E_z = -\exp\left[(r/bZ_s)\int P(\phi)d\phi\right] \times \int \exp\left[(-r/bZ_s)\int P(\phi)d\phi\right](r/b)H_r^0(r,\phi)d\phi. \quad (55)$$

Using (55) as a guide, it is a simple matter to find an E_z which satisfies the tip equation for small values of r/δ, and which approaches in a continuous manner the "standard" solution given by (53) for large values of r/δ. The condition that none of the field components become independent of ϕ as r approaches 0 can usually be satisfied by inspection. This technique, therefore, enables one, with a minimum amount of guessing, to find a *continuous* function which satisfies the correct boundary conditions and the wave equation near the tip of the wedge, and which approaches the "standard" coupled mode solution away from the tip. The transition region can be estimated from (49) since it is determined by the distance from the tip of the wedge at which the standard surface impedance condition used in most waveguide problems breaks down.

In the present example, however, E_z^e and H_r^0 do not satisfy (53) and H_r^0 has to be written in the form,

$$H_r^0 = -(j\beta/2\alpha)[J_{t-1}(\alpha r) - J_{t+1}(\alpha r)]\sin(t\phi) = H_r^A + H_r^B. \quad (56)$$

In (47), however, E_z^e has the form,

$$E_z^e = E_z^A + E_z^B \quad (57)$$

where

$$E_z^A = \frac{Z_s H_r^A \cos[(t-1)\phi]}{\cos[(t-1)\phi_1]\sin(t\phi)} \quad (58)$$

and

$$E_z^B = \frac{Z_s H_r^B \cos[(t+1)\phi]}{\cos[(t+1)\phi_1]\sin(t\phi)}, \quad (59)$$

both of which are of the form of (53). Both E_z^B and H_r^B vanish instead of becoming infinite at the wedge tip and (59) can be assumed to be valid for all values of r. Eq. (58) represents the dominant fields for small values of r/δ, and H_r^A can be used to replace H_r^0 in (53). A field E_z can then be found as before, using (55) as a guide. This function, of course, approaches E_z^A in (58) for large values of r/δ, but E_z^B as given by (59) can be added to the solution for all values of r without affecting the behavior of the solution near the tip of the wedge since E_z^B vanishes as r^{t+1}. When this is done, the coupled axial electric field in the wedge waveguide is given, on the faces of the wedge, by the simple expression,

$$E_z = \frac{-(\beta\alpha/\omega\epsilon) r J_t'(\alpha r)[(r/bZ_s) + t\cos(t\phi_1)]}{[(r/bZ_s)^2 + t^2]}, \quad (60)$$

where $b = (j\omega\epsilon/\alpha^2)$. For large values of $|r/b\ Z_s|$ this is just the surface impedance condition relating E_z to H_r^0 on the wedge surface. For small values of $|r/b\ Z_s|_s$ (60) holds for all values of ϕ simply by replacing ϕ_1 by ϕ. When this is done it is a simple matter to show that $H_r^e = -H_r^0$ and $H_\phi^e = -H_\phi^0$ as required by the analysis in Section II.

Since the present theory assumes that (50) holds for all values of r, the total radial magnetic field $H_r = H_r^0 + H_r^e$ is proportional to E_z^e in (60) on the faces of the wedge. This field differs from H_r^0 for small values of r/δ and a plot of H_r in a typical wedge waveguide is given in Fig. 3 for different values of δ/λ_0, where δ is the skin depth, λ_0 the free-space wavelength, and D the diameter of the waveguide.

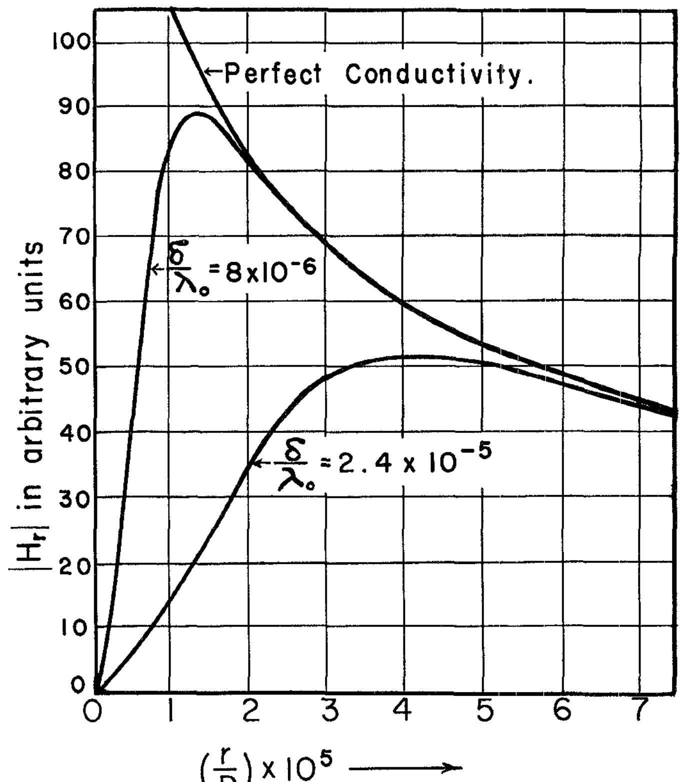

Fig. 3—The variation of the total radial magnetic field with distance from the tip of the wedge in a one-degree wedge waveguide operating in its lowest mode. D is the diameter of the waveguide, λ_0 the free-space wavelength, and δ the skin depth. In this case $\lambda_0/D = 2.0$.

V. Ohmic Losses

When (19) to (22) are solved for $\bar{L}_1$ and $\bar{\Lambda}_2$, the electric field components *within* the wedge are given, near the tip, by

$$\bar{E}_z = -[j\beta/\sigma\cos(t\pi)](\bar{r})^t\cos(t\bar{\phi}), \quad (61)$$

$$\bar{E}_r = [t/\sigma\cos(t\pi)](\bar{r})^{t-1}\cos(t\bar{\phi}), \quad (62)$$

and

$$\bar{E}_\phi = -[t/\sigma\cos(t\pi)](\bar{r})^{t-1}\sin(t\bar{\phi}), \quad (63)$$

when it is assumed that the axial magnetic field, outside of the wedge, has the form,

$$H_z = (r)^t\sin(t\phi), \quad (64)$$

near the tip of the wedge. If $|E|$ is the rms value of the electric field in volts per meter, and if $\tilde{E}$ is the complex conjugate of E, the total power loss per unit length within a sector of the wedge bounded by a small radius r_0, is given by

$$W = \int_0^{r_0}\int_{-\phi_2}^{+\phi_2}\{\sigma E\cdot\tilde{E}\}\bar{r}d\bar{r}d\bar{\phi} \text{ watts per meter.} \quad (65)$$

For small values of r_0/δ, this integrates to

$$W = t\phi_0[2\sigma\cos^2(t\pi)]^{-1}r_0^{2t} \text{ watts per meter.} \quad (66)$$

This is vanishingly small for small values of r_0, demonstrating that the power dissipated in the tip region is small. As σ becomes infinite, the power dissipated in the tip region approaches zero. The expression is valid for very small values of ϕ_0, the total wedge angle, the limiting value of ϕ_0 being determined by the breakdown of (33). For vanishingly small values of ϕ_0, however, the effect of the wedge disappears for any finite value of σ and the minimum value of t approaches unity rather than $\frac{1}{2}$.

A more practical approach to the loss problem is to integrate the real part of the complex Poynting vector $S_\phi{}^*$ over the faces of the wedge. In terms of rms field quantities, $S_\phi{}^*$ is given by

$$S_\phi{}^* = E_z\tilde{H}_r - E_r\tilde{H}_z \text{ watts per square meter.} \quad (67)$$

If the value of E_z is given by (60) (or, in general, if E_z is derived by the method outlined in Section IV) the function $S_\phi{}^*$ is integrable over the entire wedge face yielding a small flow of power into the tip region which is consistent with (66).

When this method is used to calculate the attenuation constant of a wedge waveguide, the contribution from the region of the wedge surface which is less than one skin depth from the tip is insignificantly small. When the standard, unmodified method is used, however, the contribution from this region dominates the contributions to the attenuation constant from all other regions of the waveguide surface. For an 18° wedge waveguide operating in its lowest mode at a frequency halfway between cutoff and the cutoff frequency of the next lowest mode, the contribution from this "tip region" is equal to the contribution from the rest of the wedge surface. The total attenuation constant found by the unmodified method in this case is 58 per cent larger than that found by the method described in the present paper. For a typical $\frac{1}{2}$° wedge waveguide, moreover, the unmodified method yields a contribution from the tip region which is 50 times the contribution from the rest of the wedge surface. This yields an attenuation constant which is more than 30 times as large as that found by the method described in the present paper.

Numerical values for the attenuation constants of wedge waveguides have been published elsewhere [11]. These are given as functions of wedge angle, conductivity, and frequency over the usable range of such waveguides. The attenuation constant varies slowly with the wedge angle and approaches a finite value as the wedge angle approaches zero.

VI. Conclusions

The exact field behavior near the tip of a mathematically sharp, imperfectly conducting wedge has been analyzed and it has been shown that the field components satisfy static boundary conditions in the region of the tip. It has also been shown that the only singularity which can occur in any component of the electric field at the tip of the wedge depends only on the dielectric constant and the conductivity of the wedge. Similarly it has been shown that the only singularity which can occur in any magnetic field component depends only on the permeability of the wedge. This means that highly conducting wedges with a permeability equal to that of the surrounding medium cannot support a singularity in any of the magnetic field components as the perfectly conducting wedge appears to do. The magnetic field components which are perpendicular to the axis of the wedge, however, do in many cases become very large *near* the tip of the wedge but, even if the wedge is perfectly sharp, these field components must reach a maximum value and then vanish at the tip of the wedge. The position of this maximum is established approximately by examining the point at which the coupled radial magnetic field, which occurs as a result of the finite conductivity, becomes comparable in magnitude to the radial magnetic field of the perfectly conducting solution. As the conductivity increases, the curve of H_r vs r approaches the curve of $H_r{}^0$ vs r, where $H_r{}^0$ is the radial magnetic field near a perfectly conducting wedge. Even though $H_r{}^0$ becomes infinite at the tip of the wedge, however, H_r reaches a maximum and decreases to zero at the tip for all finite values of wedge conductivity. This is very similar to the way in which a finite Fourier series approaches a discontinuous function as the number of terms taken becomes infinite.

It has also been shown that the surface impedance condition holds to within a few skin depths of the tip of a metal wedge and that, as long as the tangential magnetic field is not assumed to be that of the perfectly conducting problem, this condition yields results which are in agreement with the exact analysis based on the power series approach, when the condition is assumed to hold over the entire wedge face. Since the axial currents induced in the wedge do not become infinite but rather vanish at the tip, even when the wedge is perfectly sharp, any errors introduced by applying the surface impedance condition to the entire wedge face will be small. The fact that "real" wedges, moreover, are not perfectly sharp does not invalidate their representation by a perfectly sharp mathematical model since the "difference" region does not carry large currents.

In diffraction problems it makes very little difference whether the secondary fields are caused by "Huygen sources" *near* the tip of the wedge, or whether they are caused by conduction currents actually flowing on the wedge. In calculating ohmic losses, however, this difference is extremely important and must be taken into account in such calculations as those involved in determining the attenuation constant of wedge waveguides.

The present investigation is not a mathematically rigorous solution to any particular wedge problem, and further refinements would be needed if it were necessary to determine the fields in the tip region to a very high degree of accuracy. The ideas developed should be looked upon as a second approximation to the problem of boundary conditions on an imperfectly conducting wedge, the standard surface impedance condition being the first approximation. This approximation does, however, embody some very important features of wedge behavior and gives results for ohmic losses which agree reasonably well with those found in practice.

The radial electric field E_r, on the surface of the wedge, can be made proportional to the axial magnetic field H_z for all values of r without contradicting the Meixner edge condition. This, and conditions on other boundaries, can usually be satisfied by the introduction of additional coupled modes which satisfy known boundary conditions of either the Neumann or the Dirichlet type, and which are vanishingly small in the tip region. The problem of finding such solutions is often very difficult but once the radial magnetic field and the axial electric field are modified as outlined in Section IV, the boundary conditions are admissible in the sense that the resulting fields will satisfy the edge condition. In loss calculations, moreover, the normal component of the Poynting vector is needed only on the faces of the wedge, and the solution developed in Section IV is all that is needed.

Appendix I

The Surface Impedance Condition on the Face of a Wedge

Using Green's theorem [10], the axial electric field $\overline{E}_z(r', \phi')$ anywhere within the metal wedge shown in Fig. 4 can be expressed in terms of the normal derivative of this field on S_1 and in terms of the field and its derivatives on S_2 and S_3. If the function $G(r, \phi, r', \phi')$ satisfies

$$\nabla^2 G + (\bar{\alpha})^2 G = \delta(r - r')\delta(\phi - \phi') \tag{68}$$

where δ is the Dirac delta function and $\bar{\alpha}$ is defined in Section II following (4), then $\overline{E}_z(r', \phi')$ can be written in the form

$$\overline{E}_z(r', \phi') = \int_{S_1+S_2+S_3} \left\{ \overline{E}_z(r, \phi) \frac{\partial G}{\partial n} - \frac{\partial \overline{E}_z(r, \phi)}{\partial n} G \right\} dS. \tag{69}$$

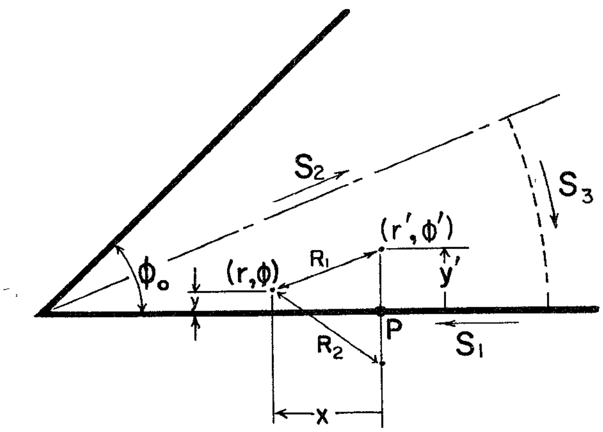

Fig. 4—The coordinate system used to establish the surface impedance condition.

Eq. (69) can be further simplified by putting

$$G(r, \phi, r', \phi') = j/4[H_0^{(2)}(\bar{\alpha}R_1) + H_0^{(2)}(\bar{\alpha}R_2)] \tag{70}$$

where R_1 is the distance between (r, ϕ) and (r', ϕ') and R_2 is the distance between (r, ϕ) and the image of (r', ϕ') in the plane S_1 as shown in Fig. 4. $H_0^{(2)}(\bar{\alpha}R)$ is a Hankel function s, of the second kind. In this case $\partial G/\partial n = 0$ on S_1 and, by symmetry for fields which are even function in ϕ in Fig. 2, $\partial \overline{E}_z/\partial n = 0$ on S_2. Eq. (69) then becomes

$$\overline{E}_z(r', \phi') = -(j/4) \int_{S_1} \{H_0^{(2)}(\bar{\alpha}R_1) + H_0^{(2)}(\bar{\alpha}R_2)\} \frac{\partial \overline{E}_z}{\partial n_1} dS_1 + (j/4) \int_{S_2} \overline{E}_z \frac{\partial}{\partial n_2} \{H_0^{(2)}(\bar{\alpha}R_1) + H_0^{(2)}(\bar{\alpha}R_2)\} dS_2. \tag{71}$$

Since $(\bar{\alpha})$ has a negative imaginary part, the integration on S_3 can be neglected, provided that S_3 is more than a few skin depths away from (r', ϕ').

Finally, letting y' approach zero in Fig. 4, R_1 approaches R_2 and (71) becomes

$$E_z(P) = -(j/2) \int_{-\infty}^{r'} H_0^{(2)}(\bar{\alpha} | x |) \frac{\partial \overline{E}_z}{\partial n} dx + (j/2) \int_{S_2} \overline{E}_z \frac{\partial}{\partial n_2} H_0^{(2)}(\bar{\alpha}R_1) dS_2. \tag{72}$$

If the constants of the wedge material are given by

$$\bar{\mu} = \mu_0,$$

and

$$\bar{\epsilon} = \epsilon_0 - j\sigma/\omega,$$

then, using (3), it is a simple matter to show that the continuity of H_r across the face of the wedge requires that, when $\phi = \phi_1$ Fig. 1,

$$\frac{\partial \overline{E}_z}{\partial n} = \frac{H_r - (\epsilon_0/\mu\sigma)(1/r)\partial E_z/\partial\phi}{-(j/\omega\mu)(1 + j\omega\epsilon_0/\sigma)} \cdot \tag{73}$$

Substituting (73) into (72) yields

$$E_z(P) = (-j/2) \int_{-\infty}^{r'} \frac{\{j\omega\mu H_r - (j\omega\epsilon/\sigma)(1/r)\partial E_z/\partial\phi\}}{(1 + j\omega\epsilon/\sigma)} \cdot H_0^{(2)}(\bar{\alpha} | x |) dx + (j/2) \int_{S_2} \overline{E}_z \frac{\partial}{\partial n_2} H_0^{(2)}(\bar{\alpha}R_1) dS_2. \tag{74}$$

For very good conductors the second term of (74) and the second term within the first integral of (74) can be neglected for all values of r'.

Appendix II

The Tip Equation

Theorem: If H_z is a given function of r and ϕ which

1) satisfies the two-dimensional Helmholtz equation,
2) satisfies the Meixner edge condition, and
3) approaches zero as r^t where $t<t_H$, the lowest eigenvalue of a magnetic type solution [see (26) and (27)],

then (37) implies (38) and any E_z satisfying (37) also satisfies the Helmholtz wave equation as r approaches zero, provided that none of the field components become independent of the angle ϕ as r approaches zero.

Eq. (37) can be written in the form,

$$\frac{\partial E_z}{\partial \phi} = \frac{\beta}{\omega\epsilon} r \frac{\partial H_z}{\partial r} + O(r^x), \tag{75}$$

as r approaches 0, where $O(r^x)$ is any function which behaves as r^x for small values of r, and $x>t_H$. Differentiating (75) with respect to r and (4) with respect to ϕ yields

$$\frac{\partial H_\phi}{\partial \phi} = \frac{-j\beta}{\alpha^2}\left[\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 H_z}{\partial \phi^2}\right] + O(r^{x+1})$$
$$= j\beta r H_z + O(r^{x-1}), \tag{76}$$

if H_z satisfies (5), the Helmholtz equation. If, moreover, H_z satisfies the Meixner edge condition, then rH_z approaches zero faster than r^{x-1} and therefore the derivative of H_ϕ with respect to ϕ behaves as r^{x-1}. If H_ϕ does not become independent of ϕ as r approaches zero, moreover, then H_ϕ also behaves as r^{x-1}, which is (38).

Differentiating (75) with respect to ϕ yields

$$\frac{\partial^2 E_z}{\partial \phi^2} = \frac{\beta}{\omega\epsilon} r \frac{\partial^2 H_z}{\partial \phi \partial r} + O(r^x). \tag{77}$$

Eq. (4), moreover, implies that as r approaches 0,

$$(-j\beta/\alpha^2 r)\frac{\partial H_z}{\partial \phi} = (j\omega\epsilon/\alpha^2)\frac{\partial E_z}{\partial r} + O(r^{x-1}), \tag{78}$$

since H_ϕ must approach 0 faster than r^{x-1}. Multiplying (78) by r, differentiating with respect to r, and substituting into (77), yields

$$\frac{\partial^2 E_z}{\partial \phi^2} = -r\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + O(r^x). \tag{79}$$

If E_z satisfies (75) and does not become independent of ϕ as r approaches 0, then E_z must approach 0 as r^t if H_z approaches 0 as r^t where $t+1>t_H-1$. This means that $(\alpha r)^2 E_z$ can be added to (79) without changing its behavior for small values of r and, therefore,

$$\frac{\partial^2 E_z}{\partial \phi^2} + r\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + (\alpha r)^2 E_z = 0 \tag{80}$$

is satisfied by E_z to within a relative error of order r^{t_H-t}.

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